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U-duality covariant membranes

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Abstract: We outline a formulation of membrane dynamics in $D = 8$ which is fully covariant under the U-duality group $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(3,\mathbb{Z})$, and encodes all interactions to fields in eight-dimensional supergravity, which is constructed through Kaluza-Klein reduction on T^3. Among the membrane degrees of freedom is an $\mathrm{SL}(2,\mathbb{R})$ doublet of world-volume 2-form potentials, whose quantised electric fluxes determine the membrane charges, and are conjectured to provide an interpretation of the variables occurring in the minimal representation of $E_{6(6)}$ which appears in the context of automorphic membranes. We solve the relevant equations for the action for a restricted class of supergravity backgrounds. Some comments are made on supersymmetry and lower dimensions.

Keywords: M-Theory, String Duality.

Contents

1. Introduction

The rôle of branes in M-theory is poorly understood and is in many respects a very puzzling subject. In string theory the distinction between fundamental and solitonic branes is clear and provides a conceptually firm ground for discussing perturbative as well as non-perturbative issues. In M-theory this distinction is washed away due to the inherently non-perturbative nature of the theory. Nevertheless it can be argued that M2-branes in some sense constitute microscopic degrees of freedom of M-theory. Matrix theory could be taken as an argument for this.

By lifting up non-perturbative results from string theory to M-theory one can get a glimpse of what might be the proper microscopic formulation of M-theory. Recently the authors of [1] suggested a way forward by trying to derive a well-established exact non-perturbative string result, namely the R^4-term obtained in a series of papers, see, e.g. [2, 3, 4], from the M2-brane in M-theory. The functions appearing as partition functions in this context present a number of interesting features and difficult problems if one wishes to obtain them from a compactified M2-brane allowed to wind the compact target space. Functions of this kind live on the product of two moduli spaces, one related to the T^3 topology of the M2-brane and one connected to the U-duality group in the lower dimension. One particular property of these automorphic functions is that their definition requires additional degrees of freedom on the M2-brane over and above those used in the conventional formulation of [5, 6]. The attempt of reference [1] turned out not to give an entirely correct result, one problem being due to incorrect counting of membrane instantons when treating them as classical membrane configurations in a saddle point approximation. The problem was solved in reference [7] by treating the winding membrane as a three-dimensional Yang-Mills theory in order to calculate the partition function. An alternative approach was put forward in reference [1, 8], where it was suggested that a proper starting point would be a membrane action already manifesting the U-duality invariance, which then demands the presence of the extra degrees of freedom mentioned above.

Due to supersymmetry on the M2-brane world-volume these additional degrees of freedom can not represent local degrees of freedom, which in fact suggests how they should be introduced. Field strengths of rank equal to the dimension of the manifold, in this case the M2 world-volume, have been used to produce exactly modes of this kind, see, e.g. [9]. In string/M theory there is an extended version of this technique where all background n-form gauge fields $A_{(n)}$ couple to their corresponding $(n-1)$-form gauge fields on the world-volume $a_{(n-1)}$ through a universal coupling $f_{(n)} = da_{(n-1)} - A_{(n)}$ [10]–[13], sometimes with some additional non-linear terms as we will see below. Using this method we will in this paper derive an action for the 2-brane in 8 dimensions which will exhibit manifest U-duality and contain the additional degrees of freedom that appear in the partition function of Pioline *et* al. [1]. That the additional degrees of freedom are related to 2-form potentials on the M2 world-volume was suggested in [8].

In this paper we will use the theory obtained by compactifying 11-dimensional supergravity on a 3-torus to discuss this problem. We perform the reduction to 8 dimensions in section 2, where we present the proper fields needed to give the Bianchi identities and duality relations in a manifest $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ covariant form. In section 3 we then apply the methods from [10]–[13] to find how the 2-brane couples to the gauge fields and scalar fields of 8 dimensional supergravity, including the two 3-form field strengths of special interest here. The formalism used gives a set of field equations and Bianchi identities which are U-duality covariant and allow for the implementation of the duality relations needed to define the theory in terms of the correct number of degrees of freedom. The discussion is performed throughout the paper in terms of bosonic fields only. In order to derive the action of a supersymmetric membrane, the same formal expressions should be used where all pullbacks to the brane are taken from a target superspace, as usual. To identify the new charges we solve the equations of motion for the world-volume 2-form potentials in section 4. In this way a (p, q) membrane action is obtained which can be compared to the spherical vector discussed in [1, 14, 15, 8]. Further comments on this connection and some conclusions are collected in section 5.

2. Maximal 8-dimensional supergravity

This section provides a derivation of the 8-dimensional supergravity obtained by dimensional reduction of $D = 11$ supergravity on T^3, with emphasis put on the transformation properties of the fields under the (continuous version of the) U-duality group in 8 dimensions, $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$. The dynamics of the bosonic sector of $D = 11$ supergravity, with vielbein $\hat{e}_M{}^A$ and 3-form potential $\hat{C}_{MNP}$ (hats are used on 11-dimensional fields), is given by the action

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{|\hat{g}|} \left[\hat{R} - \frac{1}{2 \cdot 4!} \hat{G}^2 \right] + \frac{1}{2\kappa_{11}^2} \int \frac{1}{3!}\, \hat{G} \wedge \hat{G} \wedge \hat{C} \ , \tag{2.1}$$

where $\hat{G} = d\hat{C}$ is the field strength of the 3-form $\hat{C}$. From now on we set $2\kappa_{11}^2 = 1$. Our index conventions are as follows

$$
\begin{array}{lll}
 & \text{curved} & \text{flat} \\
\text{11-dim} & M, N, P, \ldots & A, B, C, \ldots \\
\text{8-dim} & \mu, \nu, \rho, \ldots & a, b, c, \ldots \\
\text{3-dim} & m, n, p, \ldots & i, j, k, \ldots ,
\end{array}
\tag{2.2}
$$

and our metric is mostly plus. $\hat{G}$ is invariant under the gauge-transformation

$$\delta\hat{C} = d\hat{\chi} \ , \tag{2.3}$$

with $\hat{\chi}$ a 2-form. To compactify this theory on T^3 we make the following Kaluza-Klein Ansatz for the vielbein

$$\hat{e}_M{}^A = \begin{pmatrix} e_\mu{}^a & -A_\mu^{1m} e_m{}^i \\ 0 & e_m{}^i \end{pmatrix}, \qquad \hat{e}_A{}^M = \begin{pmatrix} e_a{}^\mu & e_a{}^\mu A_\mu^{1m} \\ 0 & e_i{}^m \end{pmatrix}, \tag{2.4}$$

$$dx^M = (dx^\mu, dx^m) \ , \tag{2.5}$$

where all fields are allowed to depend on x^μ only. This gives

$$\hat{e}^A = dx^M \hat{e}_M{}^A = (dx^\mu e_\mu{}^a, dx^m e_m{}^i - dx^\mu A_\mu^{1m} e_m{}^i) = (e^a, e^i - A^{1i}) = (\hat{e}^a, \hat{e}^i) \, . \tag{2.6}$$

The index 1 on A_μ^{1m} is needed to separate this vector from the one that will arise from the compactified 3-form.

Using the above information we find, after some calculations, that the 11-dimensional Einstein term gives the following contribution to the 8-dimensional action:

$$\int d^{11}x \sqrt{|\hat{g}|}\hat{R} = \int d^8x \sqrt{|g|} e^{-\varphi} \Big[R - \frac{1}{4} G_{mn} F_{ab}^{1m} F^{1n\,ab} - 2D_a (e_i{}^m \partial^a e_m{}^i) - \\ - e^{n(i}(\partial_a e_n{}^{j)}) e_i{}^m (\partial^a e_{mj}) - e_i{}^m (\partial^a e_m{}^i) e_j{}^n (\partial_a e_n{}^j) \Big],$$

where we have used the definitions

$$\sqrt{\det G_{mn}} = e^{-\varphi} \, , \qquad F_{ab}^{1m} = dA_{ab}^{1m} \, . \tag{2.7}$$

Note also that G_{mn} is the metric on T^3 constructed from the dreibein $e_m{}^i$ and that the derivatives have been written with flat indices using the 8-dimensional vielbein. We have put the parametric volume of the internal torus to one, $\int d^3y = 1$. The third term in the action can be integrated by parts

$$-2 \int d^8x \sqrt{|g|} e^{-\varphi} D_a (e_i{}^m \partial^a e_m{}^i) = 2 \int d^8x \sqrt{|g|} e^{-\varphi} (\partial_a \varphi)(\partial^a \varphi) \, , \tag{2.8}$$

where we have used the fact that

$$\mathrm{Tr}(e^{-1}\partial e) = -\partial\varphi \, . \tag{2.9}$$

Next, inserting (2.8) into (2.7) gives

$$\int d^{11}x\sqrt{|\hat{g}|}\hat{R} = \int d^8x\sqrt{|g|}e^{-\varphi}\Big[R - \frac{1}{4}G_{mn}F^{1m}_{ac}F^{1n\,ac} + (\partial_a\varphi)(\partial^a\varphi) - \\ - e^{n(i}(\partial_a e_n{}^{j)})e_j{}^m(\partial^a e_{mi})\Big]. \tag{2.10}$$

One now realises that the last term is most conveniently rewritten as

$$\begin{aligned} -\frac{1}{2}e^{ni}(\partial_a e_n{}^j)e_j{}^m(\partial^a e_{mi}) - \frac{1}{2}e^{ni}(\partial_a e_n{}^j)e_i{}^m(\partial^a e_{mj}) &= -\frac{1}{4}(G^{mn}\partial_a G_{np})(G^{pq}\partial^a G_{qm}) \\ &= -\frac{1}{4}\,\mathrm{Tr}(G^{-1}\partial G)^2\,, \end{aligned} \tag{2.11}$$

giving the action

$$\int d^{11}x\sqrt{|\hat{g}|}\hat{R} = \int d^8x\sqrt{|g|}e^{-\varphi}\Big[R + (\partial_a\varphi)(\partial^a\varphi) - \frac{1}{4}G_{mn}F^{1m}_{ab}F^{1n\,ab} - \\ - \frac{1}{4}\,\mathrm{Tr}(G^{-1}\partial G)^2\Big]. \tag{2.12}$$

With the calculation of the Ricci scalar out of the way we would now like to rescale the result to obtain an action in the Einstein frame. We perform the change of variables

$$g_{\mu\nu} = e^{\varphi/3}g^{\mathrm{E}}_{\mu\nu}\,, \qquad \mathcal{M}_{mn} = \frac{G_{mn}}{(\det G)^{1/3}}\,, \tag{2.13}$$

where the three by three matrix $\mathcal{M}_{mn}$ provides a parametrisation of the $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3,\mathbb{R})$ coset space. This gives the final form of the compactified Einstein-Hilbert term

$$\int d^{11}x\sqrt{|\hat{g}|}\hat{R} = \int d^8x\sqrt{|g^{\mathrm{E}}|}\Big[R_{\mathrm{E}} - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{4}e^{-\varphi}\mathcal{M}_{mn}F^{1m}F^{1n} - \\ - \frac{1}{4}\,\mathrm{Tr}(\partial\mathcal{M}\mathcal{M}^{-1})^2\Big]. \tag{2.14}$$

Next, we reduce the 4-form field strength. This is most easily done by expressing the forms in the flat 8-dimensional basis and the "real" (according to our Kaluza-Klein Ansatz) curved 3-dimensional basis

$$\hat{e}^m = dx^m + A^{1m}\,, \qquad A^{1m} = dx^\mu A^{1m}_\mu\,, \tag{2.15}$$

satisfying $\hat{e}^m e^i_m = \hat{e}^i$, as well as $d\hat{e}^m = dA^{1m} = F^{1m}$. This basis is invariant under transformations $\delta x^m = -\lambda^m$ of the internal torus as is seen from

$$\begin{aligned} \delta\hat{e}^m &= \delta(dx^m + dx^\mu A^{1m}_\mu) = d\delta x^m + dx^\mu(\delta A^{1m}_\mu) \\ &= -d\lambda^m + dx^\mu(\partial_\mu\lambda^m) = -d\lambda^m + d\lambda^m = 0\,, \end{aligned} \tag{2.16}$$

where we have used

$$\delta A^{1m}_\mu = \partial_\mu\lambda^m\,. \tag{2.17}$$

This is the usual way reparametrisations generate gauge transformation in a Kaluza-Klein reduction. When expanding the 11-dimensional 3-form into lower-dimensional components (8-dimensional in our case) it is convenient, and completely standard, to perform field redefinitions of the various fields to get them inert under this gauge invariance. Using our basis defined above, and the fact that $\hat{C}$ is manifestly invariant, we conclude that the fields appearing in the expansion (2.19) below, do in fact not transform either. Note that the gauge fields A^1 in $\hat{e}^m$ will not appear explicitly anywhere since when integrating out the three compact directions from the action only terms with three dx^m's will be non-zero. Therefore $\hat{e}^m$ appears as dx^m from the point of view of the action.

Let us begin by reducing the $D = 11$ relation

$$\delta\hat{C} = d\hat{\chi}\,. \tag{2.18}$$

As explained above, we expand the 11-dimensional 3-form into λ^m invariant fields as follows:

$$\hat{C} = C' + B'_m \wedge \hat{e}^m + \frac{1}{2!}A^{2p} \wedge \epsilon_{mnp}\hat{e}^m \wedge \hat{e}^n + \frac{1}{3!}a\epsilon_{mnp}\hat{e}^m \wedge \hat{e}^n \wedge \hat{e}^p\,, \tag{2.19}$$

where ϵ_{mnp} is defined to be constant. Repeating this for the gauge parameter, using the superspace convention of exterior derivatives acting from the right, we find

$$\begin{aligned} d\hat{\chi} &= d(\chi' - \chi'_m \wedge \hat{e}^m + \frac{1}{2!}\epsilon_{mnp}\chi'^{2p}\hat{e}^m \wedge \hat{e}^n) \\ &= \underbrace{d\chi' - \chi'_m \wedge F^{1m}}_{\delta C'} + \underbrace{(d\chi'_m + \epsilon_{mnp}\chi'^{2p}F^{1n})}_{\delta B'_m} \wedge \hat{e}^m + \underbrace{d\chi'^{2p}}_{\delta A'^{2p}} \wedge \epsilon_{mnp}\frac{1}{2!}\hat{e}^m \wedge \hat{e}^n\,, \end{aligned} \tag{2.20}$$

where we have also indicated which field transformations the terms are connected to. Moreover, one immediately finds the field strengths that are invariant under the above λ^m transformations as follows:

$$\begin{aligned} \hat{G} = d\hat{C} &= d(C' + B'_m \wedge \hat{e}^m + \frac{1}{2!}A^{2p} \wedge \epsilon_{mnp}\hat{e}^m \wedge \hat{e}^n + \frac{1}{3!}a\epsilon_{mnp}\hat{e}^m \wedge \hat{e}^n \wedge \hat{e}^p) \\ &= \underbrace{dC' + B'_m \wedge F^{1m}}_{G} + \underbrace{(-dB'_m + A^{2p}\epsilon_{mnp} \wedge F^{1n})}_{-H_m} \wedge \hat{e}^m + \\ &\quad + \frac{1}{2!}\underbrace{(dA^{2p} + aF^{1p})}_{F'^{2p}}\epsilon_{mnp} \wedge \hat{e}^m \wedge \hat{e}^n - \frac{1}{3!}(da)\epsilon_{mnp} \wedge \hat{e}^m \wedge \hat{e}^n \wedge \hat{e}^p\,. \end{aligned} \tag{2.21}$$

Hence, our 8-dimensional field strengths become:

$$\begin{aligned} G &= dC' + B'_m \wedge F^{1m}\,, \\ H_m &= dB'_m - \epsilon_{mnp}F^{1n} \wedge A^{2p}\,, \\ F'^{2m} &= F^{2m} + aF^{1m} = dA^{2m} + aF^{1m}\,, \end{aligned} \tag{2.22}$$

satisfying the following non-trivial Bianchi identities

$$dG = H_m F^{1m}\,, \qquad dH_m = -\epsilon_{mnp}F^{1n} \wedge F^{2p}\,, \qquad dF'^{2m} = da \wedge F^{1m}\,. \tag{2.23}$$

Next, we reduce the $\hat{G}^2$ term in the 11-dimensional action. Using (2.22) and rewriting the $\hat{G}^2$ term in terms of forms

$$S_{\hat{G}^2} = -\int \frac{1}{2}\hat{G}(*_{11}\hat{G})\,, \tag{2.24}$$

we obtain, after a short calculation, the following contribution to the 8-dimensional action:

$$\begin{aligned} S_{\hat{G}^2} = -\int d^8x\sqrt{|g_{\mathrm{E}}|}\,\Bigg[&\frac{1}{48}e^{-\varphi}G^2 + \frac{1}{12}H_m H_n \mathcal{M}^{mn} + \\ &+\frac{1}{4}e^{\varphi}\mathcal{M}_{mn}F'^{2m}F'^{2n} + \frac{1}{2}e^{2\varphi}(\partial a)^2\Bigg]\,, \end{aligned} \tag{2.25}$$

where $G^2 = G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}$. In the final result we have also switched to the Einstein metric. Note also that when reducing, the $*_{11}$ splits nicely into $(*_8)(*_3)$. The next step is to write the Einstein term together with the $\hat{G}^2$ term in a more $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ covariant way. This leads to

$$\begin{aligned} \int d^{11}x\sqrt{|\hat{g}|}\left(\hat{R} - \frac{1}{48}\hat{G}^2\right) = \int d^8x\sqrt{|g_{\mathrm{E}}|}\Bigg[&R_{\mathrm{E}} - \frac{1}{4}\,\mathrm{Tr}(\partial\mathcal{W}\mathcal{W}^{-1})^2 - \frac{1}{4}\,\mathrm{Tr}(\partial\mathcal{M}\mathcal{M}^{-1})^2 - \\ &-\frac{1}{4}\mathcal{M}_{mn}F^{rm}F^{sn}\mathcal{W}_{rs} - \frac{1}{12}H_m H_n\mathcal{M}^{mn} - \\ &-\frac{1}{48}e^{-\varphi}G^2\Bigg]\,, \end{aligned} \tag{2.26}$$

where $r, s = 1, 2$, and $\mathcal{W}$ is the following metric parametrising the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ coset:

$$\mathcal{W} = \frac{1}{\mathrm{Im}(\tau)}\begin{pmatrix} |\tau|^2 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & 1 \end{pmatrix} = e^{\varphi}\begin{pmatrix} a^2 + e^{-2\varphi} & 0 \\ a & 1 \end{pmatrix}\,, \tag{2.27}$$

with $\tau = a + ie^{-\varphi}$. Excluding the G^2 term, it is easy to see that (2.26) is invariant under the following $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(3,\mathbb{R})$ transformations:

$$\mathcal{W} \to \Lambda\mathcal{W}\Lambda^{\mathrm{T}}\,, \qquad F^m \to (\Lambda^{\mathrm{T}})^{-1}F^m\,, \qquad \Lambda \in \mathrm{SL}(2)\,, \tag{2.28}$$

$$\mathcal{M} \to R\mathcal{M}R^{\mathrm{T}}\,, \qquad H_m \to R_m{}^n H_n\,, \qquad F^m \to F^n(R^{-1})_n{}^m, \qquad R \in \mathrm{SL}(3)\,. \tag{2.29}$$

The metric in the Einstein frame is invariant under both transformations while the 4-form G requires a separate discussion which will be presented after treating the Chern-Simons term.

In writing down the Chern-Simons term we use that the terms resulting from the product must not have more than three 3-beins (or equivalently after integrating out the compact directions a form of degree eight must remain), all other terms are trivially zero. This gives

$$\begin{aligned} \int \frac{1}{6}\hat{G}\wedge\hat{G}\wedge\hat{C} = \int d^8x\frac{1}{6^3\cdot 2^4}\Big[&GGa + 8GH_m A^{2m} + 12GF'^{2m}B'_m + \\ &+8G(da)C' - 8H_m H_n B'_p\epsilon^{mnp} + 16H_m F'^{2m}C'\Big]\,, \end{aligned} \tag{2.30}$$

where all space-time indices have been suppressed (the eight indices on each term are understood as being contracted with an epsilon-tensor). In conclusion the complete action for $D = 11$ supergravity reduced to 8 dimensions is given by (2.26) and (2.30).

Up to this point we have followed closely the presentation in [16]. However, the discussion below will differ somewhat adding a couple of clarifying points to previous work.

Before we continue and reduce the 7-form we make the following observation: Looking at how the field strength H_m is defined above, we see that it is not written in an $\mathrm{SL}(2,\mathbb{R})$ invariant way. This is fixed by redefining the potential B'_m as follows:

$$B'_m = B_m - \frac{1}{2}\epsilon_{mnp}A^{1n} \wedge A^{2p}\,, \tag{2.31}$$

which implies that

$$H_m = dB_m - \frac{1}{2}\epsilon_{mnp}(F^{1n} \wedge A^{2p} - F^{2n} \wedge A^{1p}) = dB_m - \frac{1}{2}\epsilon_{mnp}\epsilon_{rs}F^{rn} \wedge A^{sp}\,, \tag{2.32}$$

where $r, s = 1, 2$. It is also convenient to redefine C' as follows:

$$C' = C - \frac{1}{3}A^{1m} \wedge B_m + \frac{1}{6}\epsilon_{mnp}A^{1m} \wedge A^{2n} \wedge A^{1p}\,, \tag{2.33}$$

which implies that

$$G = dC + \frac{2}{3}B_m \wedge F^{1m} - \frac{1}{3}A^{1m} \wedge H_m\,. \tag{2.34}$$

Note that the Bianchi identities are of course not changed by these redefinitions. It is clear, however, that the new B and C fields do transform under reparametrisations along the internal 3-torus which is unavoidable consequence of imposing manifest SL(2) covariance.

Next we are going to show how the 11-dimensional 7-form reduces to, among other things, a 4-form, denoted as G', which is dual to the 4-form G. The 11-dimensional 4- and 7-forms can be expanded as follows:

$$\begin{aligned}\hat{G} &= G - H_m \wedge \hat{e}^m + \frac{1}{2!}F'^{2p} \wedge \epsilon_{mnp}\hat{e}^m \wedge \hat{e}^n - \frac{1}{3!}da \wedge \epsilon_{mnp}\hat{e}^m \wedge \hat{e}^n \wedge \hat{e}^p\,,\\ \hat{G}_7 &= G_7 - G_{6m} \wedge \hat{e}^m + \frac{1}{2!}G_{5mn} \wedge \hat{e}^m \wedge \hat{e}^n - \frac{1}{3!}G' \wedge \epsilon_{mnp}\hat{e}^m \wedge \hat{e}^n \wedge \hat{e}^p\,.\end{aligned} \tag{2.35}$$

From $\hat{G}_7$ we are only interested in the 4-form G' (since higher tensors do not couple to membranes), which is the field strength we need to form an SL(2) doublet together with G.

We begin by deriving the Bianchi identity for G', using the Bianchi identity for $\hat{G}_7$

$$d\hat{G}_7 = \frac{1}{2}\hat{G} \wedge \hat{G}\,. \tag{2.36}$$

Next, from the above Bianchi identity and (2.35), we obtain the following Bianchi identity for G':

$$dG' = G \wedge da + H_m \wedge F'^{2m} = G \wedge da + H_m \wedge F^{2m} + H_m \wedge F^{1m}a\,. \tag{2.37}$$

We also find that the duality relation $\hat{G}_7 = *_{11}\hat{G}$ gives the following duality relation between the two 4-forms in eight dimensions:

$$G' = -e^{-\varphi}(*_8 G)\,. \tag{2.38}$$

One slight disadvantage with the way we have defined G' is that it is difficult to write the two 4-forms as an $\mathrm{SL}(2,\mathbb{R})$ doublet. To rectify this we define a new 4-form $\tilde{G}$ as

$$\tilde{G} = G' - aG = -e^{-\varphi}(*_8 G) - aG\,. \tag{2.39}$$

This implies that the Bianchi identity for $\tilde{G}$ is given by

$$d\tilde{G} = H_m \wedge F^{2m}\,. \tag{2.40}$$

Hence, we can write the Bianchi identities for the two 4-forms $\tilde{G}$ and G in the following $\mathrm{SL}(2,\mathbb{R})$ covariant form:

$$dG^r = H_m \wedge F^{rm}\,, \tag{2.41}$$

where we have defined $G^1 = G$, $G^2 = \tilde{G}$, $r = 1,2$, $F^{2m} = dA^{2m}$ and

$$G^r = dC^r + \frac{2}{3}B_m \wedge F^{rm} - \frac{1}{3}A^{rm} \wedge H_m\,. \tag{2.42}$$

Moreover, using the $\mathrm{SL}(2,\mathbb{R})$ covariant notation for the two 4-forms implies that the duality relation (2.39) can be conveniently rewritten in the following way

$$*_8 G^r = -\epsilon^{rs}G_s = -\epsilon^{rs}\mathcal{W}_{st}G^t\,, \tag{2.43}$$

where $\epsilon^{12} = 1$ and $\mathcal{W}$ is the symmetric $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2)$ matrix given above in (2.27).

Furthermore, in this $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ covariant notation the variation of the potentials is given by:

$$\begin{aligned} \delta A^{rm} &= d\chi^{rm}\,, \qquad \delta B_m = d\chi_m - \frac{1}{2}\epsilon_{mnp}\epsilon_{rs}A^{rn} \wedge d\chi^{sp}\,, \\ \delta C^r &= d\chi^r - \frac{2}{3}A^{rm} \wedge d\chi_m + \frac{1}{3}B_m \wedge d\chi^{rm} + \frac{1}{6}\epsilon_{mnp}\epsilon_{st}A^{rm} \wedge A^{sn} \wedge d\chi^{tp}\,, \end{aligned} \tag{2.44}$$

where χ^{rm} is a 0-form, χ^m a 1-form and χ^r a 2-form. The field strengths G^r, H_m and F^{rm}, are gauge invariant under the gauge transformation (2.44).

The $\mathrm{SL}(2,\mathbb{R})$-doublet we have derived above can be encoded in a slightly more elegant manner by using complex $\mathrm{SL}(2,\mathbb{R})$-zweibeins. These are defined by a complexification of the real zweibeins which are in turn extracted from the $\mathrm{SL}(2,\mathbb{R})$-metric, $\mathcal{W}$, according to

$$\mathcal{W}_{rs} = \nu_r{}^I \nu_s{}^J \delta_{IJ}\,, \tag{2.45}$$

giving us (modulo SO(2) transformations, i.e. in a certain SO(2) gauge)

$$\nu_1{}^1 = ae^{\varphi/2}, \quad \nu_1{}^2 = e^{-\varphi/2}\,, \quad \nu_2{}^1 = e^{\varphi/2}\,, \quad \nu_2{}^2 = 0\,. \tag{2.46}$$

Complexifying these, $\mathcal{U}_r = \nu_r{}^1 + i\nu_r{}^2$, then gives

$$\mathcal{U}_1 = ae^{\varphi/2} + ie^{-\varphi/2} = e^{\varphi/2}\tau\,, \tag{2.47}$$

$$\mathcal{U}_2 = e^{\varphi/2}\,. \tag{2.48}$$

The $\mathrm{SL}(2,\mathbb{R})$-indices can then be absorbed using these zweibeins giving us

$$\mathcal{G} = \mathcal{U}_r G^r = e^{\varphi/2}(aG^1 + G^2) + ie^{-\varphi/2}G^1\,, \tag{2.49}$$

$$\mathcal{F}^m = \mathcal{U}_r F^{rm} = e^{\varphi/2}(aF^1 + F^2) + ie^{-\varphi/2}F^1\,. \tag{2.50}$$

Similarly, $\mathrm{SL}(3,\mathbb{R})$-invariant fields are obtained by contracting with dreibeins $\mathcal{V}_m{}^i$ fulfilling $\mathcal{V}\mathcal{V}^T = \mathcal{M}$. Converting from SL to SO hides the "metric" information in the action and the F^2 term, for example, takes the form $\mathcal{F}^i\bar{\mathcal{F}}^i$. Bianchi identities for SL-invariant field strengths include the left-invariant Maurer-Cartan forms for $\mathcal{U}$ and $\mathcal{V}$, and the kinetic terms of scalars may be written as the square of the part of the Maurer-Cartan form outside the gauged *so* algebra. This SL-invariant notation, in addition to a certain amount of elegance, becomes necessary when we want to consider fermionic fields and supersymmetry. The $U(1)$ acting on the complex zweibeins is also the one that rotates the complex spinors in $D = 8$.

To summarise this section, we here collect some useful formulas, namely the manifest U-duality covariant Bianchi identities for the 2-, 3-, and 4-form field strengths, with $\epsilon_{12} = +1$,

$$dG^r = H_m \wedge F^{rm}\,, \qquad dH_m = -\frac{1}{2}\epsilon_{rs}\epsilon_{mnp}F^{rn} \wedge F^{sp}\,, \qquad dF^{rm} = 0\,, \tag{2.51}$$

and the 8-dimensional duality relation for the 4-form,

$$*_8 G^r = -\epsilon^{rs}G_s = -\epsilon^{rs}\mathcal{W}_{st}G^t\,. \tag{2.52}$$

3. U-duality covariant membrane dynamics

The aim of this section is to write down an action for a membrane that couples to all the fields in the supergravity background derived in the previous chapter. This is done via world-volume field strengths roughly of the form "$f = da - A$", where a is a world-volume field and A the pullback of a background field. Any background tensor field (of low enough rank to couple to a membrane) has its world-volume counterpart. This procedure was originally invented for a somewhat different purpose [17], but it soon became evident that it has several interesting features: it encodes the background coupling, and thereby the way branes may end on each other, in a covariant manner, and, in cases where branes themselves come in multiplets of a symmetry group of the supergravity, a single action encodes all charge sectors. The formalism was developed and generalised in references [10]–[13]. The last property is what is interesting to us here; it will allow us to formulate membrane dynamics in a way that accounts for the two types of membranes occurring in 8 dimensions, namely the direct reduction of the M2-brane and the winding M5-brane, carrying electric charges with respect to the two 3-form gauge fields in the supergravity theory (or, considering self-duality, electric and magnetic charge, respectively, under one of them). These charges are identified in terms of electric fluxes of an $\mathrm{SL}(2)$ doublet of 2-form potentials on the membrane. This identification allows for a direct physical interpretation of the variables occurring in non-linear realisation of the automorphic membrane group $E_{6(6)}$ [1, 14, 8, 18].

A formulation of brane dynamics where every background field has a world-volume counterpart will automatically be covariant with respect to the global symmetries of the background (ungauged) supergravity to which the brane couples, i.e. under the U-duality group. The principal form of the background coupling, "$f = da - A$" (which in the case of modified Bianchi identities in the background will be suitably modified, see below), and of the background gauge transformations $\delta A = d\Lambda$, which are accompanied by a shift in the world-volume potential, $\delta a = \Lambda$, is directly related to the general nature of world-volume fields being Goldstone modes corresponding to background "gauge symmetries", that become global symmetries for parameters taking non-zero values on the brane [19, 20].

As will be exemplified below, this procedure generically leads to a situation where the number of world-volume fields is larger than the number of physical Goldstone modes. Constraints must be placed on the fields, typically in the form of some self-duality condition. By making a general enough Ansatz for the action and demanding that the constraints are consistent with the way the background fields occur in Bianchi identities and in equations of motion determines the action. In the present case, as will be clear below, we have not succeeded in solving this problem for general backgrounds, due to a certain non-linearity of the constraints. This will not affect our general conclusions, but means that we have not strictly speaking achieved a formulation that is covariant under the full U-duality group for general $D = 8$ supergravity backgrounds. The technical details are explained later in this section.

The governing principle in writing down the field strengths is of course gauge invariance. Having modified Bianchi identities in the background will demand that we add non-trivial terms to $da - A$ to account for the way that the background potentials transform. A convenient form of the modifying terms is "world-volume potential $\times$ background field strength". A short calculation reveals

$$\begin{aligned} w^{rm} &= d\phi^{rm} - A^{rm}\,, \\ f_m &= da_m - B_m + \frac{1}{2}\varepsilon_{mnp}\varepsilon_{rs}\phi^{rn}F^{sp}\,, \qquad (3.1) \\ h^r &= db^r - C^r - \frac{1}{3}\phi^{rm}H_m - \frac{2}{3}a_m \wedge F^{rm} + \\ &\quad +\alpha w^{rm} \wedge f_m + \beta\varepsilon_{mnp}\varepsilon_{st}\mathcal{W}^{rs}\mathcal{W}_{uv}w^{tm} \wedge w^{un} \wedge w^{vp}\,, \qquad (3.2) \end{aligned}$$

These field strengths are invariant under the target space gauge transformations in (2.44) together with

$$\begin{aligned} \delta\phi^{rm} &= \chi^{rm}\,, \\ \delta a_m &= \chi_m - \frac{1}{2}\varepsilon_{mnp}\varepsilon_{rs}A^{rn}\chi^{sp}\,, \\ \delta b^r &= \chi^r - \frac{2}{3}A^{rm} \wedge \chi_m + \frac{1}{3}B_m\chi^{rm} + \frac{1}{6}\varepsilon_{mnp}\varepsilon_{st}A^{rm} \wedge A^{sn}\chi^{tp}\,. \qquad (3.3) \end{aligned}$$

Below, we will make use of the following Bianchi identities

$$dw^{rm} = -F^{rm}\,, \qquad (3.4)$$

$$df_m = -H_m + \frac{1}{2}\varepsilon_{mnp}\varepsilon_{rs}F^{rn} \wedge w^{sp}\,. \qquad (3.5)$$

The last two terms in the definition of h^r, containing the parameters α and β, yet to be determined, are obviously gauge invariant on their own accord. Their inclusion is explained below. Note that the β-term, which in the real formalism we use looks a bit complicated, in complex formalism equals the simple expression $i\beta\varepsilon_{mnp}w^m \wedge w^n \wedge \bar{w}^p$.

Counting the number of degrees of freedom gives a too high number. Apart from the five transverse scalars, a supersymmetric membrane should only have three additional bosonic degrees of freedom, which can be taken as for example the triplet of internal scalars ϕ^{1m}. The doublet of 2-forms do not contain any local degrees of freedom, but the remaining scalars and the vectors should be related to the physical scalars by some relation. This relation is a duality relation, as will be explained in a little while. The "action" we will write down initially does not have this duality relation as an equation of motion, but is consistent with it. This situation has been encountered earlier in references [12, 13] .

The actions for various branes in the type of formulation we use is always of the form "$S \sim \int \lambda(1+\Phi+h^2)$" where Φ is some polynomial function of world-volume field strengths (except the maximal ones). They are quadratic in the max-forms h. The variable λ is a scalar Lagrange multiplier. Note that there is no Wess-Zumino term; that coupling is instead reproduced by the h^2 term once the equation of motion for λ is used to eliminate h. We thus write

$$S = \int d^3\xi\sqrt{-g}\lambda\left[1 + \Phi(w,f) - \star h^r \star h^s \mathcal{W}_{rs}\right], \tag{3.6}$$

where Φ is a, yet unknown, function we aim to obtain. The method for determining its exact form is to consider consistency of the background couplings encoded in the field strengths with some duality relation. This is where the last two terms in the definition of h^r enter; they are needed for consistency of the (any) duality. We also note that we could equivalently leave them out and instead introduce terms linear in h in the action.

We begin by deriving the equations of motion from the above action. They will, due to the fact that we do not know the form of Φ, be implicit. By taking into account how the potentials enter the various field strengths, we get

$$\begin{aligned}
\phi^{rm} :\ & d\left[\lambda \star j_{rm} - 2\alpha\lambda f_m \wedge \star h^s \mathcal{W}_{sr} - 2\beta\lambda\varepsilon_{mnp}(\varepsilon_{sr}\mathcal{W}^{vs}\mathcal{W}_{ut} + 2\varepsilon_{st}\mathcal{W}^{vs}\mathcal{W}_{ur})w^{un}w^{tp} \star h^{s'}\mathcal{W}_{s'v}\right] \\
& = \lambda\left[-\frac{1}{2}\varepsilon_{mnp}\varepsilon_{rs} \star k^n F^{sp} + \frac{2}{3}H_m \star h^s \mathcal{W}_{sr} + \alpha\varepsilon_{mnp}\varepsilon_{rs}w^{tn}F^{sp} \star h^u \mathcal{W}_{ut}\right], \qquad (3.7) \\
a_m :\ & d\left[\lambda \star k^m - 2\alpha\lambda w^{rm} \star h^s \mathcal{W}_{sr}\right] + \frac{4}{3}\lambda F^{rm} \star h^s \mathcal{W}_{sr} = 0, \qquad (3.8) \\
b^r :\ & d(\lambda\mathcal{W}_{rs} \star h^s) = 0, \qquad (3.9) \\
\lambda :\ & 1 + \Phi - \star h^r \star h^s \mathcal{W}_{rs} = 0, \qquad (3.10)
\end{aligned}$$

where we have left out the equations of motion for the transverse scalars since these enter the duality discussion in a trivial manner. We have also defined the quantities

$$j_{rm} = \frac{\partial\Phi}{\partial w^{rm}}, \qquad k^m = \frac{\partial\Phi}{\partial f_m}. \tag{3.11}$$

The last equation, (3.10), is a constraint relating the square of h to the fields. It will be responsible for the non-linearity of the duality relations.

The equations of motion for b^r, (3.9), state that $\lambda\mathcal{W}_{rs}\star h^s$, which is also the electric field conjugate to b^r, is constant. Assuming that it takes integer values due to single-valuedness of wave functions [21], it can be interpreted as a doublet of membrane charges p_r.

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By inserting equation (3.9) into the equations of motion for the a's (3.8) and rewriting the l.h.s. in terms of background potentials we see that it is automatically satisfied if

$$\star k^m = 2\left(\frac{2}{3}+\alpha\right) w^{rm} \star h^s \mathcal{W}_{rs}\,. \tag{3.12}$$

If this relation holds, these equations of motion are identical to the Bianchi identities (3.4) for the scalars ϕ, i.e. the background fields enter in the same way on the right hand side. This is the first of the implicit duality relations. It can in turn be substituted into equation (3.7) to yield

$$\begin{aligned} d[\lambda \star j_{rm} - 2\alpha\lambda f_m \wedge \star h^s \mathcal{W}_{sr} \; - \; 2\beta\lambda\varepsilon_{mnp}\left(\varepsilon_{sr}\mathcal{W}^{vs}\mathcal{W}_{ut} + 2\varepsilon_{st}\mathcal{W}^{vs}\mathcal{W}_{ur}\right) w^{un} w^{tp} \star h^{s'}\mathcal{W}_{s'v}] = \\ = \lambda\left[-\frac{2}{3}\varepsilon_{mnp}\varepsilon_{rs} w^{un} F^{sp} \star h^t \mathcal{W}_{tu} + \frac{2}{3} H_m \star h^s \mathcal{W}_{sr}\right]. \end{aligned} \tag{3.13}$$

Demanding that this equation of motion in turn is automatically satisfied, using the Bianchi identities for a (3.5) and ϕ gives us the second duality relation

$$\begin{aligned} \star j_{rm} = 2\left(\alpha - \frac{1}{3}\right) f_m \star h^s \mathcal{W}_{sr} + \varepsilon_{mnp}\Bigg(\left[-2\beta + \frac{1}{3}\right]\varepsilon_{rs} \star h^{t'}\mathcal{W}_{t't} w^{sn} w^{tp} + \\ + 2\beta\varepsilon_{st} \star h^s w^{un} w^{tp}\mathcal{W}_{ur}\Bigg). \end{aligned} \tag{3.14}$$

Note that the right hand sides of the equations of motion indeed are integrable, which allows the identification of the implicitly defined "conjugate variables" j and k with the above expressions in terms of field strengths.

Thinking of Φ as some non-linear expression whose lowest terms in a power expansion in fields are proportional to w^2 and f^2, and examining the content of equations (3.12) and (3.14), we note that equation (3.12) is a non-linear duality relation between f and a certain projection of w, namely the one that "points in the same direction as $\star h^r$" (in complex notation $\mathrm{Re}(\star\bar{h}w)$). Equation (3.14), on the other hand, contains two components, of which one, its contraction with $\star h^r$, again is a non-linear duality relation, while the other one, its contraction with $\star h^t \epsilon_{tu}\mathcal{W}^{ur}$, does not contain f on the right hand side (these two components correspond to the real and imaginary parts of $\star\bar{h}\star j$). The counting of degrees of freedom tells us that the nine components of a_m and ϕ^{rm} together represent only three physical degrees of freedom. Therefore, the constraints imposed on their field strengths should effectively contain two triplets of vectors, and the necessary consistency condition (on Φ) is that this indeed happens.

At this point, we have not been able to solve the system in complete generality. Compared to situations encountered earlier, where the equations analogous to (3.12) and (3.14) have been simply a pair of duality relations, the system at hand is more complicated. The non-linearity does not reside entirely in the factor $\star h$, but appears also on the right hand side of equation (3.14). The equations are very similar to those for (p,q) 5-branes in type IIB, commented on in reference [12] and partially solved in a series expansion in reference [22]. The method of the latter of these references is useful since it indicates that a unique solution exists. The situation we have been able to handle exactly in the present case is the one where the two duality relations are equivalent and the remaining constraints are independent of these. This happens when

$$\star h^r \epsilon_{rs} w^s = 0\,, \tag{3.15}$$

i.e. when all components of the vector triplet w point in the same direction as h. This is not a property of the solution in the most general situation, however, as it turns out to put some restriction on the background fields, as will be seen below.

We will examine this restricted solution by starting from the assumption that h and w are aligned, with respect to their SL(2) indices. We then note that if $\beta = \frac{1}{6}$, the w^2-terms on the right hand side of equation (3.14) vanish. We get the relations

$$\begin{aligned} \star k^m &= 2\left(\alpha + \frac{2}{3}\right) \star h^s \mathcal{W}_{rs} w^{rm}\,, \\ \star j_{rm} &= 2\left(\alpha - \frac{1}{3}\right) \star h^s \mathcal{W}_{rs} f_m\,. \end{aligned} \tag{3.16}$$

The following calculations are simplified by the realisation that the SL(2) indices drop out completely when all fields point in one direction. We can then treat h and w as if they were single-component. Define the two vector triplets v and u by

$$w^{rm} = \frac{v^m \star h^r}{\sqrt{\star h^s \star h^t \mathcal{W}_{st}}}\,, \tag{3.17}$$

$$f_m = -\star u_m \tag{3.18}$$

(in complex language, w carries the same phase as $\star h$ and v is its modulus: $\star h = |\star h| e^{i\chi}$, $w^m = v^m e^{i\chi}$). The duality relations between f and w now turn into algebraic relations between u and v expressed as

$$\begin{aligned} \frac{\partial \Phi}{\partial v} &= 2\left(\alpha + \frac{2}{3}\right)\sqrt{1+\Phi}\, u\,, \\ \frac{\partial \Phi}{\partial u} &= 2\left(\alpha - \frac{1}{3}\right)\sqrt{1+\Phi}\, v\,, \end{aligned} \tag{3.19}$$

where the equation of motion for the Lagrange multiplier λ has been used in order to replace h by the positive square-root. The essential consistency check is that these two equations must contain exactly the same information. This is a condition on the function Φ. It is easily checked (for example by the first terms in a series expansion) that the parameter α must take the value $-1/6$. Then the numerical factors on the right hand side of equation (3.19) become 1 and -1. It is possible to simplify the equations further: by the substitution $\rho = \sqrt{1+\Phi}$ they turn into

$$\begin{aligned} \frac{\partial \rho}{\partial v} &= u\,, \\ \frac{\partial \rho}{\partial u} &= -v\,. \end{aligned} \tag{3.20}$$

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One has to remember, however, that Φ should not contain any constant (independent of u and v) term, which rules out trivial solutions to equation (3.20) like $\rho = \frac{1}{2}(v^2 - u^2)$, $v = u$. In fact, it is Φ, not ρ, that turns out to be polynomial.

The problem of finding the "right" Φ can now be formulated as follows: We wish to obtain a Φ which when inserted into the two duality relations above equates them. We have not been able to prove strictly that this requirement fixes Φ uniquely, although a general series expansion in powers of u and v, using an implementation of the methods below in Mathematica, indicates that this is the case. Instead of pursuing that kind of general analysis, we will make an Ansatz for the duality relation directly by comparison with the ordinary M2-brane wrapped on T^3. Let us therefore turn to the dynamics of the membrane to see how such a duality arises.

The internal scalars of the $M2$-brane compactified on T^3 enter the action according to

$$S = \int d^3\xi \sqrt{-\det G}\,, \tag{3.21}$$

with

$$\det G_{\alpha\beta} = \det(g_{\alpha\beta} + v^i_\alpha v_{i\beta})\,. \tag{3.22}$$

The field strength v of the internal scalars is identical to the pullback of the internal vielbein $\hat{e}$ of equation (2.4). In what will follow we do not write out the indices on v, viewing it instead as a 3×3-matrix. This means not caring about the signature of the world-volume metric; it turns out to be irrelevant for these algebraic considerations. If we define the invariants[1]

$$\begin{aligned}
T_2 &= \operatorname{Tr} v^2\,, && (3.23)\\
T_4 &= \frac{1}{2}\left(\operatorname{Tr} v^4 - (\operatorname{Tr} v^2)^2\right), && (3.24)\\
T_6 &= \frac{1}{3}\operatorname{Tr} v^6 - \frac{1}{2}\operatorname{Tr} v^2 \operatorname{Tr} v^4 + \frac{1}{6}(\operatorname{Tr} v^2)^3 = -(\det v)^2\,, && (3.25)
\end{aligned}$$

we see that the 3×3 matrix v obeys the Cayley-Hamilton identity

$$v^6 = T_6 + T_4 v^2 + T_2 v^4\,, \tag{3.26}$$

which implies that $\{g, v, v^2, v^3, v^4, v^5\}$ is a suitable basis to express our duality in. The duality, now in a form that relates vector to vector, is then given by

$$u = \frac{\partial \mathcal{L}}{\partial v} = -\sqrt{-\det G}(G^{-1}v)\,, \tag{3.27}$$

which we will demand to be consistent with an action of the form (3.6). Therefore we make an Ansatz for Φ which is an arbitrary polynomial of degree six in v and u (implying that the variation of our Ansatz yields all the independent terms). Such a polynomial will

[1] We use a shorthand notation, where we, as mentioned, suppress the internal metric, and in addition omit transposition of matrices, so that, e.g. $\operatorname{Tr} v^2$ means $\operatorname{Tr}(v^t v)$. The Cayley-Hamilton equation is most easily derived in a frame where v is diagonal.

have 27 independent coefficients which are determined by demanding equivalence between the duality relation (3.27) and each of the two relations (3.19). It is a priori not at all obvious that a Φ exists that fulfills these relations. Seen as a series expansion in u and v, they contain an infinite number of equations for a finite number of constants. Therefore it is very gratifying to verify that a solution exists. It was obtained by implementing the Cayley-Hamilton relation in Mathematica, and is of the form.[2]

$$\Phi = \frac{1}{2}\Big[\operatorname{Tr} v^2 - \operatorname{Tr} u^2 + \frac{1}{2}(\operatorname{Tr}(uv))^2 - \operatorname{Tr}(u^2v^2) - \\ - \frac{1}{3}\operatorname{Tr} v^6 + \frac{1}{2}\operatorname{Tr} v^4 \operatorname{Tr} v^2 - \frac{1}{6}(\operatorname{Tr} v^2)^3\Big]. \tag{3.28}$$

It is worth mentioning that in the case we have solved, both equations (3.12) and (3.14) (given Φ) are non-linear equations involving both variables, while equation (3.27) represents a solution of u in terms of v. One may try to solve for v in terms of u, but this turns out to amount to solving a fifth order equation. Even with a solution $v(u)$ at hand, one should not try to use u (i.e. f) alone to describe membrane dynamics, since f obeys a modified Bianchi identity involving the scalars.

In the restricted solution presented above we have assumed that w^r and h^r point in the same direction. The action is thus not covariant under the full SL(2) group. Since we effectively have only one w and one h out of the doublets, the actual symmetry is, in a suitable basis, generated by one of the two generators of $SL(2,\mathbb{Z})$, acting as "$\tau \to \tau + 1$". The reason for this is that the original input in our solution, namely $\star h^r \epsilon_{rs} w^s = 0$, only is a valid solution in certain backgrounds. Acting on this equation (multiplied by λ) with an exterior derivative yields the condition

$$p_r \varepsilon^{rs} \mathcal{W}_{st} F^t = 0 \tag{3.29}$$

(where p_r are the charges that arise when solving the equations of motion for b_r, as discussed in the earlier in this section), putting restrictions on the background. In addition, this restriction depends on which charge sector we are considering. For such a background, equation (3.15) and the calculation following it constitute a valid solution to the relations (3.12) and (3.14). However, due to the p-dependence of (3.29) it is not meaningful to call this a U-duality invariant formulation in such a background. In order to claim U-duality covariance, we would need to restrict to a smaller class of backgrounds contained in (3.29), namely $F = 0$. In such a background the action (3.6) with Φ given by (3.28) gives a U-duality covariant formulation of membrane dynamics[3].

In more general backgrounds, we would need to go back to the equations (3.12) and (3.14) and make a more general Ansatz for Φ. It is quite clear from the structure

[2] One linear combination of the constants in the Ansatz remains undetermined. However, the function it multiplies vanishes identically in Φ and its variations when the duality relation holds. We have chosen the simplest version of Φ.

[3] The function Φ is given in terms of v, not w. Although the definition of v in equation (3.17) involves $\star h$ explicitly through it phase, this dependence cancels when each pair of v's is replaced by a w and a $\bar{w}$, or equivalently, by $\mathcal{W}_{rs} w^r w^s$. This is of course necessary in order for the equation of motion for the 2-forms to be unaffected.

of the non-linear terms we earlier chose to discard from (3.14) that Φ will then contain odd as well as even powers of fields. A series expansion and implementation of the Cayley-Hamilton identities using e.g. Mathematica could probably give the correct result, but we have so far not been able to solve the equations.

An alternative way to finding the membrane dynamics would be to consider κ-symmetry. One then considers the supermembrane action given by the same formal expression, but with the background fields being pullbacks of superspace tensor fields. For an unknown Φ, the κ-variation is undetermined, but just as for the equation of motion, assuming a duality relation turns them into explicit expressions in terms of world-volume fields. One then uses the background dimension 0 and 1/2 values of the background tensor fields, makes an Ansatz for the half-rank projector on κ (these tend to be especially simple in the present formalism). The κ-invariance together with consistency of the projection would then give the same duality relation as derived above. This is not surprising, since the non-linear duality and generalised chirality implied by κ-symmetry are intimately linked together. We have not performed this calculation, but are convinced that it will yield the same information, as was the case in references [11, 12, 13].

4. Elimination of the 2-forms

In order to eliminate the 2-forms and reformulate the dynamics while retaining U-duality covariance, we first note that the equation of motion (3.9) for b^r implies

$$\lambda \star h^s \mathcal{W}_{rs} = p_r = \text{constant}\,. \tag{4.1}$$

This expression defines the two constants $p_1 = p$ and $p_2 = q$, which in complete analogy with the string case [21, 10, 11] have to be integers in order for the quantum mechanical wave-function to be single valued [21]. Inserting, for example, these p_r into our implicit duality relations then results in

$$\lambda \star k^m = w^{sm} p_s\,, \tag{4.2}$$

$$\lambda \star j_{rm} = -f_m p_r\,, \tag{4.3}$$

when restricted to the simpler set of backgrounds with $F = 0$. Note that for constant (p, q) we have thus effectively broken the $\mathrm{SL}(2,\mathbb{Z})$-invariance, although in an $\mathrm{SL}(2,\mathbb{Z})$-covariant looking manner. This means that we have restricted our action, which previously described the entire $\mathrm{SL}(2,\mathbb{Z})$-orbit of (p, q)-membranes, to describe only one such membrane. Next we wish to derive the action for such a membrane.

Define the field strength

$$\tilde{w}^m \equiv p_r w^{rm} = \sqrt{p^2}\, v^m\,, \tag{4.4}$$

of a field $\tilde{\phi}^m = p_r \phi^{rm}$ where the second equality follows from (3.17). Here, $\sqrt{p^2} = \sqrt{p_r p_s \mathcal{W}^{rs}} = e^{\varphi/2}|p - q\tau|$, and it is important for the derivation of the action and for the interpretation of $\tilde{\phi}$ as an internal coordinate that we use a linear combination of the w^{rm} with constant coefficients that at the same time is proportional to v^m. Using the duality relation (3.27), the pair of Bianchi identities (3.4) and (3.5) can be turned into a pair of equation of motion and Bianchi identity for the scalar $\tilde{\phi}$. Indeed this is the definition of our duality. So by inserting (3.27) into (3.5) (using the relations between the field strengths given above), the Bianchi identity for f, we can eliminate a and obtain the equation of motion for $\tilde{\phi}$

$$d\left[\sqrt{-\det G}\star\left(G^{-1}\frac{\tilde{w}_m}{\sqrt{p^2}}\right)\right] = -H_m\,. \tag{4.5}$$

At the same time, the expression for the metric G in terms of $\tilde{w}$ is

$$G = g + \frac{\tilde{w}\tilde{w}^t}{p^2}\,, \tag{4.6}$$

so integrating the equation of motion gives a (p,q)-membrane action with a non-trivial scalar dependence in the tension:

$$S_{(p,q)} = -\int d^3\xi\sqrt{p^2}\sqrt{-\det G} + \int\left(\left(C^r - \frac{1}{3}A^{rm}\wedge B_m\right)p_r + \tilde{w}^m\wedge B_m\right), \tag{4.7}$$

Here we have used the relation $\delta\det M = \det M\,\mathrm{Tr}(M^{-1}\delta M)$ for the first term of the action and added a (p,q)-covariant 3-form, following from the equations of motion for the coordinates, to the WZ-term.

The kinetic term in the action tells us that the (p,q) membrane tension, in the Einstein frame, is proportional to $\sqrt{p^2}$. We would now like to compare this kinetic term to the one in equation (3.13) of reference [8], namely

$$S_1 = \frac{\sqrt{\det[ZZ^t + (y^2+x_0^2)\mathbb{I}]}}{y^2+x_0^2} = \sqrt{y^2+x_0^2}\sqrt{\det\left[\mathbb{I} + \frac{ZZ^t}{y^2+x_0^2}\right]}. \tag{4.8}$$

Here, Z is the winding matrix, which after fixing world-volume reparametrisations become identical to $\tilde{w}$, while g on the euclidean brane becomes equal to the identity matrix $\mathbb{I}$. Identifying $y^2+x_0^2$ with p^2 (at $\tau = i$) means that the kinetic terms are equal. The variables x_0 and y is the pair of membrane charges p_r. The corresponding actions also agree at arbitrary values of τ (a more general form is also given in reference [8]). The "action" of reference [8] also contains a Wess-Zumino term (denoted S_2), which upon compactification of a $D = 11$ membrane is obtained as the WZ term of a membrane completely winding T^3 and thus coupling linearly to the axion (a membrane instanton). In reference [8] it arises from algebraic considerations, by demanding invariance under the larger group E_6. We have not considered these aspects here. We comment on this in the Conclusions.

We end this section with a direct check of the dilaton dependence of the (p,q) membrane tension. According to equation (4.7), the tension for a (1,0)-membrane divided by the tension for a (0,1)-membrane, for $a = 0$, is given by $\mathrm{T}_{(1,0)}/\mathrm{T}_{(0,1)} = g$.

Next, using results from section 2 we can give another argument why $\mathrm{T}_{(1,0)}/\mathrm{T}_{(0,1)} = g$. We start by using that in 11 dimensions the Newton constant is defined as

$$2\kappa_{11}^2 = (2\pi)^8\ell_{11}^9\,, \tag{4.9}$$

where ℓ_{11} is the 11-dimensional Planck length. In 8 dimensions we instead have defined the following Newton constant:

$$2\kappa_8^2 = (2\pi)^5 \ell_8^6 \,, \tag{4.10}$$

where ℓ_8 is the 8-dimensional Planck length. These constants are related as follows:

$$2\kappa_{11}^2 = \mathrm{vol}(T^3) 2\kappa_8^2 \,. \tag{4.11}$$

Moreover, in the reduction above we used that $\sqrt{\det G_{mn}} = e^{-\varphi}$. Hence, this implies that

$$\mathrm{vol}(T^3) = (2\pi)^3 g^{-1} \ell_{11}^3 \,, \tag{4.12}$$

where g is the closed string coupling constant. This means that using (4.11) and (4.12), we obtain the following relation between the Planck length in 11 dimensions and the Planck length in 8 dimensions:

$$\ell_{11} = g^{-1/6} \ell_8 \,. \tag{4.13}$$

Next, from the membrane in 11 dimensions we obtain a (1,0)-membrane with the same tension in units of the 11-dimensional Planck length. However, expressed in units of the 8-dimensional Planck length we instead obtain, using (4.13)

$$\mathrm{T}_{(1,0)} = \frac{1}{(2\pi)^2 \ell_{11}^3} = \frac{1}{(2\pi)^2 g^{-1/2} \ell_8^3} \,. \tag{4.14}$$

Furthermore, from the M5-brane we obtain, by wrapping it on the 3-torus, a $(0,1)$-membrane in 8 dimensions. The tension is given by

$$\mathrm{T}_{(0,1)} = \mathrm{T}_{\mathrm{M5}} \times \mathrm{vol}(T^3) = \frac{\mathrm{vol}(T^3)}{(2\pi)^5 \ell_{11}^6} = \frac{1}{(2\pi)^2 g^{\frac{1}{2}} \ell_8^3} \,. \tag{4.15}$$

Hence, using (4.14) and (4.15) we get that $\mathrm{T}_{(1,0)}/\mathrm{T}_{(0,1)} = g$, which is the same as we obtained above.

5. Conclusions

In this paper we have derived a membrane world-volume action in 8 dimensions. The main result, explained in detail in section 3, is that the action is given by

$$S = \int d^3\xi \sqrt{-g} \lambda \left[1 + \Phi(w,f) - \star h^r \star h^s \mathcal{W}_{rs}\right] , \tag{5.1}$$

where w, f and h are field strengths for scalars, vectors and 2-forms on the world-volume, including coupling to supergravity background potentials, and where Φ, for the restricted class of background with vanishing 2-form field strengths, is given by equation (3.28). The coupling to background fields is consistent with the duality to be imposed on the world-volume fields in addition to the equations of motion encoded by the action.

8 is the highest dimensionality where the U-duality group is "non-trivial", due to the emergence of an SL(2) factor relating metric and tensorial degrees of freedom. This is therefore a suitable arena for trying to understand the origin of the spherical vectors appearing in the work on membrane partition functions by Pioline et al. [1, 8], in particular the role played by the extra degrees of freedom on the membrane that appear to be required by their construction.

The method used here produces a theory that couples the membrane to all fields in the supergravity background in a manifestly U-duality symmetric way. The important property of this method is that it associates an n-form field strength on the world-volume to every n-form gauge potential in the background. Thus, in 8 dimensions the world sheet theory will automatically contain two 2-form potentials whose field equations can be solved producing two integration parameters which can be identified as the charges appearing in the partition functions of Pioline et al. [1, 8]. This construction can in principle be generalized to lower dimensions and bigger U-duality groups, although we expect that the problems we had with solving the equations for general backgrounds will persist or get worse.

This conclusion is reached by comparing our action for the (p, q) membrane and the classical action (see for example equation (3.13) in the recent paper [8] or (4.8) above) for the automorphic membrane. This points strongly towards interpreting the "extra" dimensions of the configuration space as the charges of the covariant membrane. The SL(2)-charges in our action enter in the exactly same way as the "extra variables" in the classical automorphic membrane action at least for the part of the spherical vector of the automorphic membrane that comes out of our analysis, namely the kinetic term S_1.

This lends some support for the conjecture in [1, 8] that the extra variables in the minimal representation based on E_6 are really related to membrane charges coming from 2-form potentials on the world-volume. In our particular case the (p, q) membranes originate from both M2 and M5 branes in 11 dimensions which follows from the fact that they couple to 2-forms coming from both the 3-form and the 6-form potential in 11 dimensions as explained in detail above. Note that the completely winding configurations counted in the work of Pioline et al. are membrane instantons which do not arise from M5 branes winding the compact dimensions, at least not in the standard way. This is related to the fact that the second part, the Wess-Zumino type term S_2 in the spherical vector is not reproduced in our work. Further evidence for this interpretation can be found by repeating the argument in lower dimensions. In $D = 7$, with U-duality group $\mathrm{SL}(5, \mathbb{R})$, membrane charges come in the fundamental representation, and can, from an 11-dimensional perspective, be thought of as a membrane together with four types of M5-branes, winding the four T^3 cycles in the T^4. The counting agrees with the five "extra" variables taking part in the non-linear realisation of E_7, thus lending further support to the conjecture in [1, 8].

Although the observations made here makes the properties of membranes a bit less mysterious from an algebraic point of view, the understanding of the membrane as a "fundamental" microscopic building block of M-theory remains unclear. The analysis made in this paper is purely classical. It does not show that the proposal of reference [8] is correct, but it rather elucidates what it amounts to, namely a sum over (p, q) membrane charge sectors.

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